MOTION IIT-JEE | AIEEE CBSE | SAT | NTSE OLYMPIADS

Nurturing potential through education

MATHEMATICS

TARGET IIT JEE

BINOMIAL THEOREM

THEORY AND EXERCISE BOOKLET

CONTENTS

394 - Rajeev Gandhi Nagar Kota, Ph. No. 0744-2209671, 93141-87482, 93527-21564

IVRS No. 0744-2439051, 0744-2439052, 0744-2439053 www.motioniitjee.com, email-info@motioniitjee.com

JEE Syllabus :

Binomial theorem for a positive integral index, properties of binomial coefficients

A. BINOMIAL THEOREM

The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as BINOMIAL THEOREM. If x, y $\in$ R and n $\in$ N, then

$$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots\ldots.. + {}^nC_rx^{n-r}y^r + \ldots\ldots + {}^nC_ny^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r .$$

This theorem can be proved by induction.

Observations :

(a) The number of terms in the expansion is (n + 1) i.e. one more than the index.

(b) The sum of the indices of x & y in each term is n.

(c) The binomial coefficients of the terms (${}^nC_0$, ${}^nC_1$.........) equidistant from the beginning and the end are equal.

Ex.1 The value of $\dfrac{(18^3 + 7^3 + 3.18.7.25)}{3^6 + 6.243.2 + 15.81.4 + 20.27.8 + 15.9.16 + 6.3.32 + 64}$ is

Sol. The numerator is of the form $a^3 + b^3 + 3ab\,(a + b) = (a + b)^3$ where a = 18, and b = 7

$\therefore$ Nr = $(18 + 7)^3 = (25)^3$. Denominator can be written as

$$3^6 + {}^6C_1.3^5.2^1 + {}^6C_2.3^4.2^2 + {}^6C_3.3^3.2^3 + {}^6C_4 3^2.2^4 + {}^6C_5 3.2^5 + {}^6C_6 2^6 = (3 + 2)^6 = 5^6 = (25)^3 \therefore \frac{Nr}{Dr} = \frac{(25)^3}{(25)^3} = 1$$

B. IMPORTANT TERMS IN THE BINOMIAL EXPANSION ARE

(a) GENERAL TERM : The general term or the (r + 1)th term in the expansion of $(x + y)^n$ is given by

$$T_{r+1} = {}^nC_r x^{n-r}.\, y^r$$

Ex.2 Find : **(a)** The coefficient of x^7 in the expansion of $\left(ax^2 + \dfrac{1}{bx}\right)^{11}$

(b) The coefficient of x^{-7} in the expansion of $\left(ax - \dfrac{1}{bx^2}\right)^{11}$

Also, find the relation between a and b, so that these coefficients are equal.

Sol. **(a)** In the expansion of $\left(ax^2 + \dfrac{1}{bx}\right)^{11}$, the general terms is $T_{r+1} = {}^{11}C_r (ax^2)^{11-r}\left(\dfrac{1}{bx}\right)^r = {}^{11}C_r . \dfrac{a^{11-r}}{b^r} . x^{22-3r}$

putting 22 – 3r = 7 $\Rightarrow$ 3r = 15 $\Rightarrow$ r = 5 $\therefore T_6 = {}^{11}C_5 \dfrac{a^6}{b^5}.x^7$

Hence the coefficient of x^7 in $\left(ax^2 + \dfrac{1}{bx}\right)^{11}$ is ${}^{11}C_5\, a^6 b^{-5}$.

(b) In the expansion of $\left(ax-\frac{1}{bx^2}\right)^{11}$, general terms is $T_{r+1} = {}^{11}C_r(ax)^{11-r}\left(\frac{-1}{bx^2}\right)^r = (-1)^r\ {}^{11}C_r\frac{a^{11-r}}{b^r}.x^{11-3r}$

putting 11 – 3r = – 7 $\Rightarrow$ 3r = 18 $\Rightarrow$ r = 6 $\therefore$ $T_7 = (-1)^6.{}^{11}C_6\frac{a^5}{b^6}.x^{-7}$

Hence the coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6a^5b^{-6}$

Also given coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ = coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$

$\Rightarrow$ ${}^{11}C_5a^6b^{-5} = {}^{11}C_6a^5b^{-6}$ $\Rightarrow$ ab = 1 ($\therefore$ ${}^{11}C_5 = {}^{11}C_6$). Which is a required relation between a and b.

Ex.3 Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Sol. The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is $T_{r+1} = {}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r\ 3^{\frac{1000-r}{2}}2^{\frac{r}{2}}$

The above term will be rational if exponent of 3 and 2 are integers. i.e. $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ...1000}. Hence, number of rational terms is 501

(b) MIDDLE TERM : The middle term(s) in the expansion of $(x + y)^n$ is (are)

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2}.\ x^{n/2}.\ y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

Ex.4 Find the middle term in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$

Sol. The number of terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e. $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+3}{2}\right)$th two middle terms. They are given by T_5 and T_6

$\therefore$ $T_5 = T_{4+1} = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4 = {}^9C_43^5x^5.\frac{x^{12}}{6^4} = \frac{9.8.7.6}{1.2.3.4}.\frac{3^5}{2^4.3^4}x^{17} = \frac{189}{8}x^{17}$

and $T_6 = T_{5+1} = {}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5 = -{}^9C_43^4.x^4\frac{x^{15}}{6^5} = \frac{-9.8.7.6}{1.2.3.4}.\frac{3^4}{2^5.3^5}x^{19} = -\frac{21}{16}x^{19}$

(c) TERM INDEPENDENT OF x :

Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

Ex.5 The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is

Sol. General term in the expansion is ${}^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10}\cdot\frac{3^{5-r}}{2^{\frac{10-r}{2}}}$

For constant term, $\frac{3r}{2}=10 \Rightarrow r=\frac{20}{3}$ which is not an integer. Therefore, there will be no constant term.

(d) NUMERICALLY GREATEST TERM : To find the greatest term in the expansion of $(x + a)^n$.

We have $(x + a)^n = x^n\left(1+\frac{a}{x}\right)^n$; therefore, since x^n multiplies every term in $\left(1+\frac{a}{x}\right)^n$, it will be sufficient to find the greatest term in this later expansion. Let the T_r and T_{r+1} be the r^{th} and $(r+1)^{th}$ terms in the expansion of $\left(1+\frac{a}{x}\right)^n$ then $\frac{T_{r+1}}{T_r}=\frac{{}^nC_r\left(\frac{a}{x}\right)^r}{{}^nC_{r-1}\left(\frac{a}{x}\right)^{r-1}}=\frac{n-r+1}{r}\frac{a}{x}$. Let numerically, T_{r+1} be the greatest term in the above expansion. Then $T_{r+1} \ge T_r \Rightarrow \frac{T_{r+1}}{T_r}\ge 1 \Rightarrow \frac{n-r+1}{r}\left|\frac{a}{x}\right| \ge 1 \Rightarrow r \le \frac{(n+1)}{\left(\left|\frac{x}{a}\right|+1\right)}$

Substituting values of n and x, we get $r \le m + f$ or $r \le m$ where m is a positive integer and f is fraction such that $0 < f < 1$. In the first case T_{m+1} is the greatest term, while in the second case T_m and T_{m+1} are the greatest terms and both are equal.

Ex.6 Find numerically the greatest term in the expansion of $(3 - 5x)^{11}$ when $x = 1/5$

Sol. Since $(3 - 5x)^{11} = 3^{11}\left(1-\frac{5x}{3}\right)^{11}$. Now in the expansion of $\left(1-\frac{5x}{3}\right)^{11}$,

we have $\frac{T_{r+1}}{T_r}=\frac{(11-r+1)}{r}\left|-\frac{5x}{3}\right|=\left(\frac{12-r}{r}\right)\left|-\frac{5}{3}\times\frac{1}{5}\right| = \left(\frac{12-r}{r}\right)\left(\frac{1}{3}\right) = \left(\frac{12-r}{3r}\right) \quad \left(\because x=\frac{1}{5}\right)$

$\therefore \frac{T_{r+1}}{T_r}\ge 1 \quad \Rightarrow \frac{12-r}{3r}\ge 1 \quad \Rightarrow \quad 4r \le 12 \quad \Rightarrow \quad r \le 3 \quad \therefore \quad r = 2, 3$

so, the greatest terms are T_{2+1} and T_{3+1}. $\quad \therefore$ Greatest term (when r = 2)

$= 3^{11}\,|T_{2+1}| = 3^{11}\left|{}^{11}C_2\left(-\frac{5}{3}x\right)^2\right| = 3^{11}\left|{}^{11}C_2\left(-\frac{5}{3}\times\frac{1}{5}\right)^2\right| = 3^{11}\left|\frac{11.10}{1.2}\times\frac{1}{9}\right| = 55\times 3^9 \quad \left(\because x=\frac{1}{5}\right)$

and greatest term (when r = 3) $= 3^{11}\,|T_{3+1}| = 3^{11}\left|{}^{11}C_3\left(-\frac{5}{3}x\right)^3\right| = 3^{11}\left|{}^{11}C_3\left(-\frac{5}{3}\times\frac{1}{5}\right)^3\right| = 3^{11}\left|\frac{11.10.9}{1.2.3}\times\frac{-1}{27}\right| = 55\times 3^9$

From above we say that the value of both greatest terms are equal.

C. If $(\sqrt{A}+B)^n = I + f$, where I & n are positive integers, n being odd and 0 < f < 1, then (I + f). f = K^n where A – B^2 = K > 0 & $\sqrt{A} - B < 1$. If n is an even integer, then (I + f) (1 – f) = k^n

Ex.7 If $(6\sqrt{6}+14)^{2n+1}$ = [N] + F and F = N – [N] ; where [∗] denotes greatest integer, then NF is equal to

Sol. Since $(6\sqrt{6}+14)^{2n+1}$ = [N] + F. Let us assume that f = $(6\sqrt{6}-14)^{2n+1}$; where $0 \le f < 1$.

Now, [N] + F – f = $(6\sqrt{6}+14)^{2n+1} - (6\sqrt{6}-14)^{2n+1} = 2\left[{}^{2n+1}C_1(6\sqrt{6})^{2n}(14) + {}^{2n+1}C_3(6\sqrt{6})^{2n-2}(14)^3 + ...\right]$

⇒ [N] + F – f = even integer.

Now 0 < F < 1 and 0 < f < 1 so –1 < F – f < 1 and F – f is an integer so it can only be zero

Thus NF = $(6\sqrt{6}+14)^{2n+1}\ (6\sqrt{6}-14)^{2n+1}$ = 20^{2n+1}.

D. SOME RESULTS ON BINOMIAL COEFFICIENTS

(a) $C_0 + C_1 + C_2 + ... + C_n = 2^n$

(b) $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

(c) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n = \dfrac{(2n)!}{n!n!}$

(d) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + ... + C_{n-r}C_n = \dfrac{(2n)!}{(n+r)(n-r)!}$

Remember : (2n) ! = 2^n. n! [1.3.5........(2n – 1)]

Ex.8 If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time represents by : $\sum\limits_{0\le i<j\le n}\sum C_iC_j$ is equal to $2^{2n-1} - \dfrac{2n!}{2.n!.n!}$

Sol. Since $(C_0 + C_1 + C_2 + ... + C_{n-1} + C_n)^2 = C_0^2 + C_1^2 + C_2^2 + ... + C_{n-1}^2 + C_n^2 + ...+$

$2(C_0C_1 + C_0C_2 + C_0C_3 + ... + C_0C_n + C_1C_2 + C_1C_3 + C_1C_n + C_2C_3 + C_2C_4 + ... + C_2C_n + ... + C_{n-1}C_n)$

⇒ $(2^n)^2 = {}^{2n}C_n + 2\sum\limits_{0\le i<j\le n}\sum C_iC_j$. Hence $\sum\limits_{0\le i<j\le n}\sum C_iC_j = 2^{2n-1} - \dfrac{2n!}{2.n!.n!}$

Ex.9 If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then prove that $\sum\limits_{0\le i<j\le n}\sum (C_i + C_j)^2 = (n-1)\,{}^{2n}C_n + 2^{2n}$

Sol. L.H.S $\sum\limits_{0\le i<j\le n}\sum (C_i + C_j)^2 = (C_0 + C_1)^2 + (C_0 + C_2)^2 + ... + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + ... + (C_1 + C_n)^2$
$+ (C_2 + C_3)^2 + (C_2 + C_4)^2 + ... + (C_2 + C_n)^2 + ... + (C_{n-1} + C_n)^2$

$= n(C_0^2 + C_1^2 + C_2^2 + ... + C_n^2) + 2\sum\limits_{0\le i<j\le n}\sum C_iC_j = n.{}^{2n}C_n + 2.\left\{2^{2n-1} - \dfrac{2n!}{2.n!n!}\right\}$ {from Ex. 8}

$= n.{}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n-1).{}^{2n}C_n + 2^{2n}$ = R.H.S.

E. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES

If $n \in Q$, then $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... \infty$ provided $|x| < 1$.

Note :

(i) When the index n is a positive integer the number of terms in the expansion of $(1 + x)^n$ is finite i.e. $(n + 1)$ & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1 + x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) Following expansion should be remembered ($|x| < 1$)

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - ....\infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + ...\infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ...\infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...\infty$

(iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convenient to expand in powers of $1/x$, which then will be small.

F. APPROXIMATIONS

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + ...$$

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its squares and higher powers may be neglected then $(1 + x)^n = 1 + nx$, approximately, This is an approximate value of $(1 + x)^n$

Ex.10 If x is so small such that its square and higher powers may be neglected then find the approximate value of $\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}}$

Sol. $$\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1 - \frac{3}{2}x + 1 - \frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}} = \frac{1}{2}\left(2 - \frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(2 - \frac{19}{6}x\right)\left(1 - \frac{x}{8}\right)$$

$$= \frac{1}{2}\left(2 - \frac{x}{4} - \frac{19}{6}x\right) = 1 - \frac{x}{8} - \frac{19}{12}x = 1 - \frac{41}{24}x$$

Ex.11 The value of cube root of 1001 upto five decimal places is

Sol. $(1001)^{1/3} = (1000+1)^{1/3} = 10\left(1+\frac{1}{1000}\right)^{1/3} = 10\left\{1+\frac{1}{3}.\frac{1}{1000}+\frac{1/3(1/3-1)}{2!}\frac{1}{1000^2}+......\right\}$

= 10 {1 + 0.0003333 – 0.00000011 + ...} = 10.00333

Ex.12 The sum of $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+........\infty$ is

Sol. Comparing with 1 + nx + $\frac{n(n-1)}{2!}x^2$ + ... $\Rightarrow$ nx = 1/4 ...(i)

& $\frac{n(n-1)x^2}{2!} = \frac{1.3}{4.8}$ or $\frac{nx(nx-x)}{2!} = \frac{3}{32} \Rightarrow \frac{1}{4}\left(\frac{1}{4}-x\right) = \frac{3}{16} \Rightarrow \left(\frac{1}{4}-x\right) = \frac{3}{4} \Rightarrow x = \frac{1}{4}-\frac{3}{4} = -\frac{1}{2}$ ...(ii) {by (i)}

putting the value of x in (i) $\Rightarrow$ $n(-1/2) = 1/4$ $\Rightarrow$ n = $-\frac{1}{2}$

$\therefore$ sum of series = $(1 + x)^n = (1 - 1/2)^{-1/2} = (1/2)^{-1/2} = \sqrt{2}$

G. EXPONENTIAL SERIES

(a) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(b) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm.**

(c) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+......\infty$; where x may be any real or complex number & $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(d) $a^x = 1+\frac{x}{1!}\ell na+\frac{x^2}{2!}\ell n^2 a+\frac{x^3}{3!}\ell n^3 a+.........\infty$ where a > 0

(e) $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+......\infty$

H. LOGARITHMIC SERIES

(a) $\ell n\,(1 + x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+........\infty$ where $-1 < x \le 1$

(b) $\ell n\,(1 - x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+........\infty$ where $-1 \le x < 1$

Remember : (i) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+........\infty = \ell n\, 2$ **(ii)** $e^{\ell n\, x} = x$ **(iii)** $\ell n 2 = 0.693$ **(iv)** $\ell n\, 10 = 2.303$